VOLUME 76 SEPARATE No. 33

PROCEEDINGS

AMERICAN SOCIETY OF CIVIL ENGINEERS

SEPTEMBER, 1950

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

STRENGTH OF I-BEAMS IN COMBINED BENDING AND TORSION

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Headquarters of the Society
33 W. 39th St.
New York 18, N.Y.

PRICE $0.50 PER COPY

Published at Prince and Lemon Streets, Lancaster, Pa., by the American Society of Civil Engineers. Editorial and General Offices at 33 West Thirty-ninth Street, New York 18, N. Y. Reprints from this publication may be made on condition that the full title of paper, name of author, page reference, and date of publication by the Society are given.

Proceedings—Vol. 76 September, 1950 Separate No. 33

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Founded November 5, 1852

PAPERS

STRENGTH OF I-BEAMS IN COMBINED BENDING AND TORSION

By Basil Sourochnikoff[1]

Synopsis

In the design of a structural steel framework it is considered good practice to avoid subjecting I-beams to the combined effect of bending and torsion, because the torsional resistance of I-beams is usually low. It is often easy to determine a layout in which torsion is either entirely eliminated or greatly reduced. Sometimes, however, this is impracticable or impossible, and cases exist where beams carry eccentric loads and are therefore stressed in both bending and torsion. Laterally unsupported building spandrels offer a common example. Also, it is often impossible to avoid eccentricity of loads on beams in industrial structures because of the extreme complexity of the equipment they are designed to support. In these, and numerous other cases, the designer is called on to investigate stresses in I-beams loaded in combined bending and torsion.

Much has been written on the theory of torsion, but there appears to be little information on the design of I-beams in combined bending and torsion, apart from the method presented in the "Manual of Steel Construction."[2] In this method, the flange stresses due to vertical bending and those due to torsional bending are computed separately and superimposed to obtain the resultant bending stress. Although this method is probably the most complete of the various existing methods, one may question what allowable stress is to be used, and whether or not the allowable stress for torsional bending should be the same as that for vertical bending.

In this paper an attempt is made to develop allowable stress relations taking into account the interaction between the bending moment and the torque.

Note.—Written comments are invited for publication; the last discussion should be submitted by March 1, 1951.

[1] Eng. Dept., E. I. du Pont de Nemours & Co., Wilmington, Del.

[2] "Manual of Steel Construction," Bethlehem Steel Co., Bethlehem, Pa., 1934, p. 279 et seq.

The method given in the "Manual of Steel Construction" is followed closely and extended to include the influence of the deflections on the bending and torsional moments.

Notation

Letter symbols in this paper are defined where they first appear, in the text or illustrations, and are assembled alphabetically in the Appendix, for the guidance of readers and discussers.

Introduction

It is assumed that the beam rests on simple supports and that the ends are restrained against twist. The flanges, when considered as horizontal beams, are assumed to be simply supported at the ends. Therefore, there is no flange bending moment at the supports. The loads are parallel to the initial plane of the web of the beam. Their point of application is eccentric with respect to the centroid of the cross section by distances e_1 and e_2 in the directions normal to the web and parallel to the web, respectively. The deflections are assumed to be small compared to the original eccentricities, and the torque, although not negligible, is assumed to be small compared to the bending moment. This last condition amounts to assuming that the eccentricities are small compared to the span of the beam. It is thought that these assumptions represent the case most frequently encountered in practice. Obviously, other assumptions may be made as to the fixity of the cross section at the supports, which would lead to relations similar to the ones developed in this study except for the value of coefficients. If the loads were acting in the plane of the web ($e_1 = 0$), they would produce a bending moment about an axis normal to this plane. This moment will be termed "primary" bending moment. Because the loads are eccentric, a typical cross section of the beam twists through an angle β relative to its original position. In this case, the primary bending moment has a component in the lateral direction which produces a lateral deflection. Therefore, there is both a twist and a lateral deflection even before the loads have reached the critical value producing lateral instability.[3] The assumption that the torque is small compared to the bending moment leads to the conclusion that the lateral bending moment M_y is equal to the product of the primary bending moment M_x by the angle of twist, $M_y = \beta M_x$. If the torque were not small, a second term would have to be added to the expression for the lateral bending moment.[3]

As the beam deflects laterally and twists, the actual eccentricities are changed. As an illustration, consider the case of a beam under a concentrated load at midspan (Fig. 1). If the midsection of the beam has twisted by β_o and deflected laterally by y_o, the actual eccentricity of the load with respect to the supports becomes $e_1 + e_2 \beta_o + y_o$, and the eccentricity with respect to the centroid of the cross section at midspan becomes $e_1 + e_2 \beta_o$. Therefore the torque, both at the supports and at midspan, has increased as a result of the

[3] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, p. 240, paragraph 46.

deflection and the twist. Thus, although the lateral deflection and the twist do not change the primary bending moment to a great extent, they produce a lateral bending moment and increase the torque.

In the following sections, a detailed analysis is made of two cases of loading: (1) A concentrated load at midspan and (2) a uniformly distributed load. First, however, a preliminary case, involving torque alone, must be investigated.

1. Beams Subjected to Torque Alone

It is assumed that the torque is applied in the form of couples of forces in the planes of cross sections, thereby producing torsion and lateral bending of flanges due to torsion, but no bending of the beam as a whole. It is assumed,

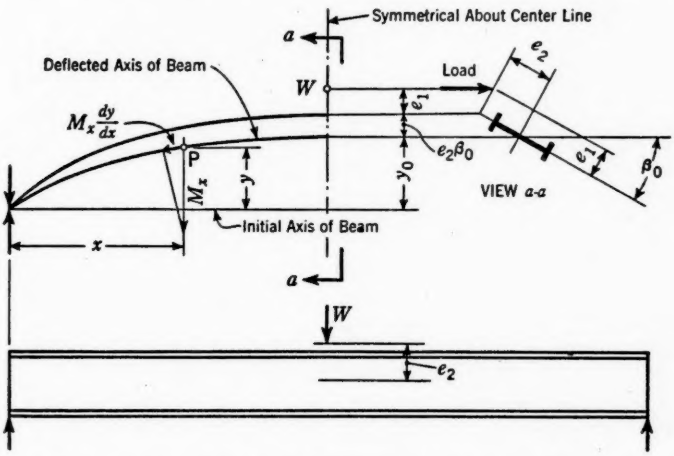

FIG. 1.—DISTORTION OF BEAM SUBJECTED TO A LOAD W AT MIDSPAN

also, that the value of the torque is not affected by the twist it produces. This preliminary case helps to simplify the study of a more general case analyzed in Section 2, where it is assumed that the beam is subjected to the combined effect of bending and torsion, and that the values of moments are affected by the deformations they cause.

The differential equation for the angle of twist ψ is:[4,5]

$$\frac{d\psi}{dx} - a^2 \frac{d^3\psi}{dx^3} = \frac{T_x}{E_s \kappa} \qquad (1)$$

in which[2] a is a torsional bending constant $\left(= \frac{d}{2}\sqrt{\frac{E\, I_y}{E_s\, \kappa}} \right)$; T_x is the torque at the point of abscissa; E is the modulus of elasticity in tension; E_s is the modulus

[4] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, p. 257, paragraph 49.

[5] "Structural Beams in Torsion," by Inge Lyse and Bruce G. Johnston, *Transactions*, ASCE, Vol. 101, 1936, p. 868 et seq.

of elasticity in shear; d is the distance between flange centroids; and κ is a torsional constant. In the various cases analyzed herein, the torque T_x is symmetrical with respect to the midspan of the beam, and therefore $\frac{d\psi}{dx} = 0$ for $x = \frac{l}{2}$. The ends of the beam are restrained against twist, but there is no restraining bending moment in the planes of the flanges at the ends. Therefore, $\psi = 0$ and $\frac{d^2\psi}{dx^2} = 0$ for $x = 0$. Thus, the boundary conditions are: For $x = 0$,

$$\psi = 0 \qquad (2a)$$

and

$$\frac{d^2\psi}{dx^2} = 0 \qquad (2b)$$

For $x = \frac{l}{2}$,

$$\frac{d\psi}{dx} = 0 \qquad (3)$$

The general solution of Eq. 1 is[4,5]

$$\psi = A_1 \sinh\frac{x}{a} + B_1 \cosh\frac{x}{a} + C_1 + X_1 \qquad (4)$$

in which X_1 is a particular solution depending on the shape of the torque. The values of the integration constants A_1, B_1, and C_1 are determined from the conditions expressed in Eqs. 2 and 3. Assuming that the lateral moment of inertia of each flange is $I_y/2$, the flange bending moment is $-\frac{E\,I_y}{4}\,d\,\frac{d^2\psi}{dx^2}$ and the section modulus is $\frac{0.5\,I_y}{0.5\,b} = \frac{I_y}{b}$. Therefore, the flange stress f_f due to lateral flange bending is

$$f_f = -\frac{E\,I_y\,d}{4}\,\frac{b}{I_y}\,\frac{d^2\psi}{dx^2} = -\frac{E\,I_y\,d^2}{4\,E_s\,\kappa} \times \frac{b\,E_s\,\kappa}{I_y\,d}\,\frac{d^2\psi}{dx^2} = -\,a\,B\,E_s\,\kappa\,\frac{d^2\psi}{dx^2} \qquad (5)$$

The flange shear is

$$H = -\frac{E\,I_y}{2}\,\frac{d}{2}\,\frac{d^3\psi}{dx^3} = -\frac{a^2}{d}\,E_s\,\kappa\,\frac{d^3\psi}{dx^3} \qquad (6)$$

The parts of the applied torque resisted by the torsional constant T' and by the lateral flange bending T'' are, respectively,

$$T' = E_s\,\kappa\,\frac{d\psi}{dx} \qquad (7a)$$

and

$$T'' = -\frac{E\,I_y\,d^2}{4}\,\frac{d^3\psi}{dx^3} \qquad (7b)$$

Deflections and Stresses for Several Types of Torque Curves.—The mathematical expressions for several types of torque (see Fig. 2) are as follows: For case 1,

$$T_x = T_o \text{ (a constant)} \qquad (8a)$$

for case 2,

$$T_x = T_o \left(1 - \frac{2x}{l}\right) \qquad (8b)$$

for case 3,

$$T_x = T_o \left[1 - \left(\frac{2x}{l}\right)^2\right] \qquad (8c)$$

and, for case 4,

$$T_x = T_o \cos\frac{\pi x}{l} \qquad (8d)$$

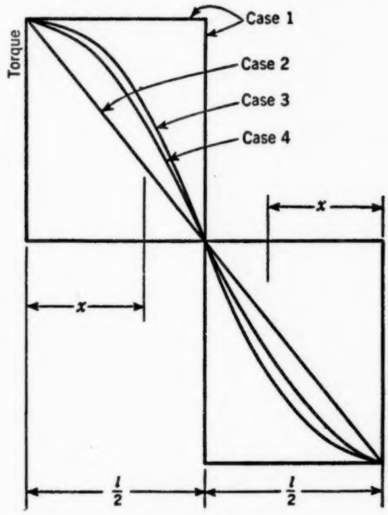

FIG. 2

The particular solutions X_1 are: For case 1,

$$X_1 = \frac{T_o l}{2 E_s \kappa}\left(\frac{2x}{l}\right) \qquad (9a)$$

for case 2,

$$X_1 = \frac{T_o l}{2 E_s \kappa}\left[\left(\frac{2x}{l}\right) - \frac{1}{2}\left(\frac{2x}{l}\right)^2\right] \qquad (9b)$$

for case 3,

$$X_1 = \frac{T_o l}{2 E_s \kappa}\left\{-\frac{1}{3}\left(\frac{2x}{l}\right)^3 + \left[1 - 2\left(\frac{2a}{l}\right)^2\right]\left(\frac{2x}{l}\right)\right\} \qquad (9c)$$

and, for case 4,

$$X_1 = \frac{T_o l}{\pi E_s \kappa \left(1 + \frac{\pi^2 a^2}{l^2}\right)} \sin\frac{\pi x}{l} \qquad (9d)$$

After determining the integration constants by the conditions defined as Eqs. 2 and 3, the values of the angle of twist become: For case 1,

$$\psi = T_o C\left[-\frac{\sinh\frac{x}{a}}{\cosh\frac{x}{a}} + \frac{l}{2a}\left(\frac{2x}{l}\right)\right] \qquad (10a)$$

for case 2,

$$\psi = T_o C\left\{\frac{2a}{l}\left(\tanh\frac{l}{2a}\sinh\frac{x}{a} - \cosh\frac{x}{a} + 1\right) + \frac{l}{2a}\left[\left(\frac{2x}{l}\right) - \frac{1}{2}\left(\frac{2x}{l}\right)^2\right]\right\} \qquad (10b)$$

for case 3,

$$\psi = T_o C \left(2 \left(\frac{2a}{l}\right)^2 \frac{\sinh \frac{x}{a}}{\cosh \frac{l}{2a}} + \frac{l}{2a} \left\{ -\frac{1}{3}\left(\frac{2x}{l}\right)^3 + \left[1 - 2\left(\frac{2a}{l}\right)^2\right]\left(\frac{2x}{l}\right) \right\} \right) \quad \text{(10c)}$$

and, for case 4,

$$\psi = T_o C \frac{1}{\frac{\pi a}{l}\left[1 + \left(\frac{\pi a}{l}\right)^2\right]} \sin \frac{\pi x}{l} \quad \text{(10d)}$$

in which[2] $C = \dfrac{a}{E_s \kappa}$ is the torsional twist constant. The maximum angle of twist occurs at midspan and may be expressed as

$$\psi_o = C R T_o \quad \text{(11)}$$

in which the quantity R is defined as follows: For case 1,

$$R = R_c = -\tanh \frac{l}{2a} + \frac{l}{2a} \quad \text{(12a)}$$

for case 2,

$$R = R_u = \frac{2a}{l}\left(\frac{1}{\cosh \frac{l}{2a}} - 1\right) + \frac{1}{2}\frac{l}{2a} \quad \text{(12b)}$$

for case 3,

$$R = R_p = 2\left(\frac{l}{2a}\right)\left[\left(\frac{2a}{l}\right)^3 \tanh \frac{l}{2a} + \frac{1}{3} - \left(\frac{2a}{l}\right)^2\right] \quad \text{(12c)}$$

and, for case 4,

$$R = R_s = \frac{1}{\frac{\pi a}{l}\left[1 + \left(\frac{\pi a}{l}\right)^2\right]} \quad \text{(12d)}$$

The variations of R with $\dfrac{l}{2a}$ are shown in Fig. 3. The shape of the curve ψ (that is, the value of the ratio ψ/ψ_o) is given in Table 1 for various values of

TABLE 1.—VALUES OF ψ/ψ_o

$\frac{2x}{l}$	$\frac{l}{2a} = 1.0$				$\frac{l}{2a} = 2.0$				$\frac{l}{2a} = 3.0$			
	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
0.25	0.365	0.390	0.375	0.384	0.350	0.393	0.379	0.384	0.334	0.422	0.376	0.384
0.50	0.686	0.713	0.703	0.707	0.665	0.717	0.705	0.707	0.641	0.765	0.703	0.707
0.75	0.918	0.925	0.915	0.925	0.905	0.925	0.925	0.925	0.893	0.985	0.921	0.925
1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

$\frac{l}{2a}$ and $\frac{2x}{l}$. It is seen that all the curves ψ/ψ_o are nearly parabolic and that the shape varies little either with $\frac{l}{2a}$ or with the form of the torque. The actual ordinates of the curves ψ, however, vary considerably, as evidenced by Fig. 3.

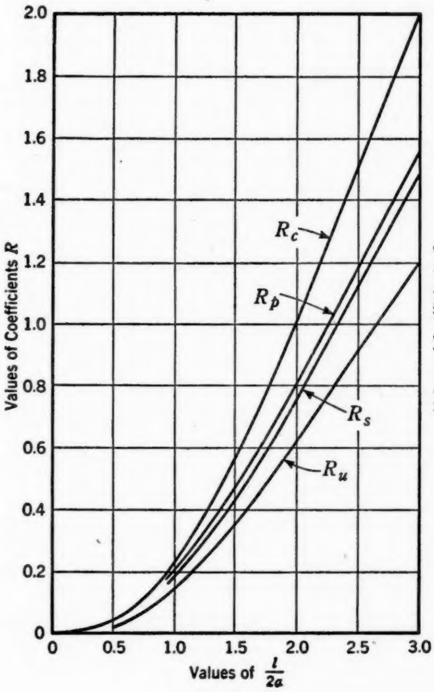

Fig. 3

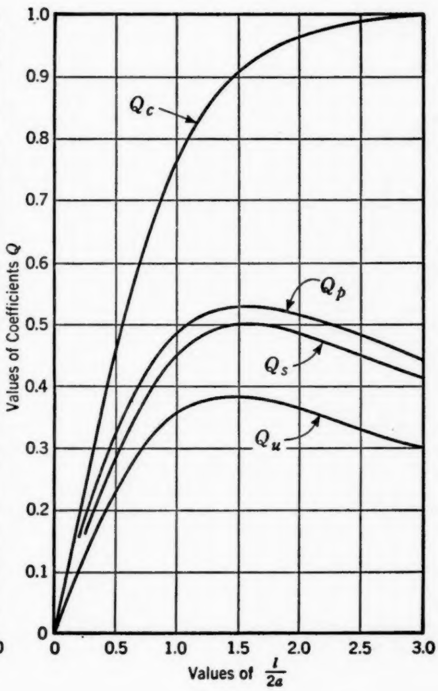

Fig. 4

The flange stress f_f is determined from Eqs. 5 and 10. Its value is found to be: For case 1,

$$f_f = B\,T_o \frac{\sinh x/a}{\cosh l/(2\,a)} \quad \text{(13}a\text{)}$$

for case 2,

$$f_f = B\,T_o \left(\frac{2\,a}{l}\right)\left(\tanh \frac{l}{2\,a} \sinh \frac{x}{a} - \cosh \frac{x}{a} + 1\right) \quad \text{(13}b\text{)}$$

for case 3,

$$f_f = 2\,B\,T_o \left(\frac{2\,a}{l}\right)\left(\frac{2\,a}{l} \frac{\sinh \frac{x}{a}}{\cosh \frac{l}{2\,a}} - \frac{2\,x}{l}\right) \quad \text{(13}c\text{)}$$

and, for case 4,

$$f_f = B\,T_o \frac{\pi\,a}{l} \frac{1}{1 + \left(\frac{\pi\,a}{l}\right)^2} \sin \frac{\pi\,x}{l} \quad \text{(13}d\text{)}$$

in which B is a torsional stress constant,[2] $\frac{a\,b}{I_y\,d}$. The maximum flange stress occurs at midspan. It may be expressed as

$$f_f = Q\,B\,T_o \qquad (14)$$

in which for case 1,

$$Q = Q_c = \tanh\frac{l}{2\,a} \qquad (15a)$$

for case 2,

$$Q = Q_u = \frac{2\,a}{l}\left[1 - \frac{1}{\cosh l/(2\,a)}\right] \qquad (15b)$$

for case 3,

$$Q = Q_p = 2\left(\frac{2\,a}{l}\right)\left[\left(\frac{2\,a}{l}\right)\tanh\frac{l}{2\,a} - 1\right] \qquad (15c)$$

and, for case 4,

$$Q = Q_s = \frac{\pi\,a}{l}\,\frac{1}{1 + \left(\frac{\pi\,a}{l}\right)^2} \qquad (15d)$$

The variation of Q with $\frac{l}{2\,a}$ is shown in Fig. 4.

A more detailed study of the torque resisted by the torsional constant shows that, in all cases except case 1, the effect of the shearing stress caused by the bending of flanges decreases as the span increases—that is, for large spans, practically all the torque is taken up by the torsional rigidity. This phenomenon is consistent with the fact that the flange stress coefficients Q decrease in value after having reached a maximum for a certain value of $\frac{l}{2\,a}$, since the effect of the flange bending decreases. In case 1, the applied torque is always taken up entirely by flange bending at midspan, regardless of the value of $\frac{l}{2\,a}$. Consistently, the coefficient Q_c increases with $\frac{l}{2\,a}$ and tends toward a finite value.

The results for case 1 are the same as those given in the "Manual of Steel Construction."[2] The results for case 2 are different. The similar case in the manual was apparently based on a different set of end conditions. Cases 3 and 4 are not found in the "Manual of Steel Construction."

2. Deflections of Beams Under Combined Effect of Bending and Torsion

An analysis of two cases of loading is presented in this section: (1) A concentrated load W at midspan and (2) a uniform load w per unit length of span. The assumptions are the same as those listed in the "Introduction."

It will be made apparent subsequently that the torque curves fall between those considered in Section 1. Therefore, β is nearly parabolic. Too much refinement in the determination of the values of β does not seem to be war-

ranted as the actual angle between any section and the sections at the supports should include the initial degree to which the beam is "out of plane." The latter variable is always present and its magnitude is unknown. It will be assumed, therefore, that β is parabolic—that is,

$$\beta = \beta_o \left[2\left(\frac{2x}{l}\right) - \left(\frac{2x}{l}\right)^2\right] \quad \text{(16)}$$

The rotation β produces a moment M_y in the lateral direction which is, for a concentrated load,

$$M_y = \beta\, M_x = \frac{W\,l}{4}\,\frac{2\,x}{l}\,\beta_o \left[2\left(\frac{2x}{l}\right) - \left(\frac{2x}{l}\right)^2\right]$$

$$= \frac{W\,l}{4}\,\beta_o \left[2\left(\frac{2x}{l}\right)^2 - \left(\frac{2x}{l}\right)^3\right] \quad \text{(17a)}$$

and, for a uniform load,

$$M_y = \beta\, M_x = \frac{w\,l^2}{8}\left[2\left(\frac{2x}{l}\right) - \left(\frac{2x}{l}\right)^2\right]\beta_o \left[2\left(\frac{2x}{l}\right) - \left(\frac{2x}{l}\right)^2\right]$$

$$= \frac{w\,l^2}{8}\,\beta_o \left[4\left(\frac{2x}{l}\right)^2 - 4\left(\frac{2x}{l}\right)^3 + \left(\frac{2x}{l}\right)^4\right] \quad \text{(17b)}$$

The lateral deflection is determined by $-\,E\,I_y\,\frac{d^2u}{dx^2} = M_y$, by double integration of the moment. The boundary conditions are $y = 0$ for $x = 0$, and $\frac{dy}{dx} = 0$ for $x = \frac{l}{2}$. Thus, for a concentrated load,

$$y = -\,\frac{W\,l^3}{16\,E\,I_y}\,\beta_o \left[\frac{1}{6}\left(\frac{2x}{l}\right)^4 - \frac{1}{20}\left(\frac{2x}{l}\right)^5 - \frac{5}{12}\left(\frac{2x}{l}\right)\right] \quad \text{(18a)}$$

and, for a uniform load,

$$y = -\,\frac{w\,l^4}{32\,E\,I_y}\,\beta_o \left[\frac{1}{3}\left(\frac{2x}{l}\right)^4 - \frac{1}{5}\left(\frac{2x}{l}\right)^5 + \frac{1}{30}\left(\frac{2x}{l}\right)^6 - \frac{8}{15}\left(\frac{2x}{l}\right)\right] \quad \text{(18b)}$$

The maximum deflection occurs at midspan in both cases: For a concentrated load,

$$y_o = 0.0187\,\frac{W\,l^3}{E\,I_y}\,\beta_o \quad \text{(19a)}$$

and, for a uniform load,

$$y_o = 0.0115\,\frac{w\,l^4}{E\,I_y}\,\beta_o \quad \text{(19b)}$$

In the case of the concentrated load, it is evident from Fig. 1 that the torque at any point p, of abscissa x, is

$$T_x = \frac{W}{2}\,(e_1 + e_2\,\beta_o + y_o - y) + M_x\,\frac{dy}{dx} \quad \text{(20a)}$$

with $M_x = \frac{W}{4}\left(\frac{2x}{l}\right)$. Using this value of M_x, with Eqs. 18*a* and 19*a* for y, $\frac{dy}{dx}$, and y_o, the torque becomes

$$T_x = \frac{W}{2}\left[e_1 + e_2\beta_o\right] + \frac{W^2 l^3}{32 E I_y}\beta_o\left[\frac{3}{10} - \frac{1}{2}\left(\frac{2x}{l}\right)^4 + \frac{1}{5}\left(\frac{2x}{l}\right)^5\right] \quad \text{..(20b)}$$

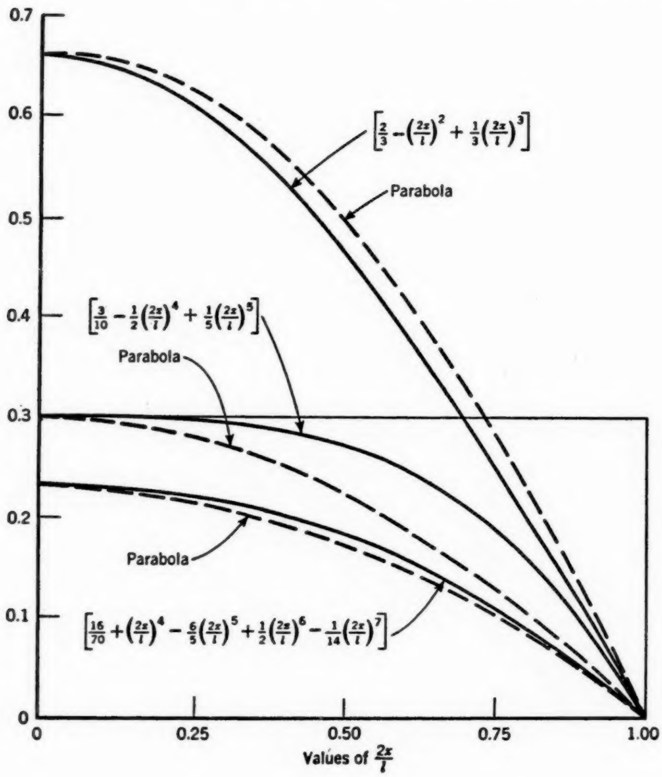

FIG. 5

The values of $\left[\frac{3}{10} - \frac{1}{2}\left(\frac{2x}{l}\right)^4 + \frac{1}{5}\left(\frac{2x}{l}\right)^5\right]$ are plotted in Fig. 5. Since the curve is quite convex and the lateral deflection was assumed to be small compared to the eccentricity e_1, there will be only a small error (on the safe side) if the torque is considered to be constant and equal to $\frac{W}{2}\left[e_1 + \left(e_2 + \frac{3}{160}\frac{W l^3}{E I_y}\right)\beta_o\right]$. On this assumption, the maximum angle of twist at midspan can be computed as follows: $\beta_o = C R_c \left[\frac{W}{2}(e_1 + e_2\beta_o)\right.$

$+ \frac{3}{320} \frac{W^2 l^3}{E I_y} \beta_o \Big]$; from which

$$\beta_o = \frac{\psi_o}{1 - \frac{\psi_o}{e_1}\left(e_2 + 0.0188 \frac{W l^3}{E I_y}\right)} \quad \text{.............. (21)}$$

Eq. 21 gives the maximum angle of twist under combined bending and torsion, computed on the assumption that the effect of the deflections on the moments is not negligible. The quantity ψ_o is the maximum angle of twist which would have been obtained if the torque alone were acting and the deflections were negligible—that is,

$$\psi_o = C R_c \frac{W e_1}{2} \quad \text{...................... (22)}$$

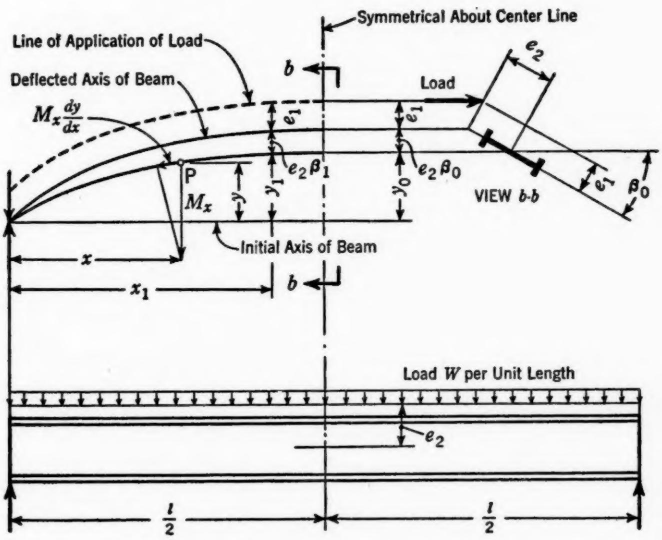

FIG. 6.—HALF-PLAN VIEW OF DEFLECTED BEAM

in which C is a torsional twist constant,[2] $\frac{a}{E_s \kappa}$. Considering now the case of the uniform load, it is evident from Fig. 6 that the torque at any point p of abscissa x is $T_x = \int_x^{l/2} w\,(e_1 + e_2\,\beta_1 + y_1 - y)\,dx + M_x \frac{dy}{dx}$; or

$$T_x = \frac{w\,l}{2} e_1 \left(1 - \frac{2\,x}{l}\right) + w\,e_2 \int_x^{l/2} \beta_1\,dx$$

$$+ w \int_o^{l/2} y_1\,dx - w\,y \frac{l}{2}\left(1 - \frac{2\,x}{l}\right) + M_x \frac{dy}{dx} \quad \text{.. (23a)}$$

Using Eqs. 18b and 19b and noting that $M_x = \frac{w\,l^2}{8}\left[2\left(\frac{2\,x}{l}\right) - \left(\frac{2\,x}{l}\right)^2\right]$, Eq. 23$a$ may be written as

$$T_x = \frac{w\,l}{2}\,e_1\left(1 - \frac{2\,x}{l}\right) + w\,e_2\,\beta_o\,\frac{l}{2}\left[\frac{2}{3} - \left(\frac{2\,x}{l}\right)^2 + \frac{1}{3}\left(\frac{2\,x}{l}\right)^3\right] + \frac{w^2\,l^5}{64\,E\,I_y}$$
$$\times\,\beta_o\left[\frac{16}{70} + \left(\frac{2\,x}{l}\right)^4 - \frac{6}{5}\left(\frac{2\,x}{l}\right)^5 + \frac{1}{2}\left(\frac{2\,x}{l}\right)^6 - \frac{1}{14}\left(\frac{2\,x}{l}\right)^7\right] \quad \text{..(23}b\text{)}$$

The values of $\left[\frac{2}{3} - \left(\frac{2\,x}{l}\right)^2 + \frac{1}{3}\left(\frac{2\,x}{l}\right)^3\right]$ and $\left[\frac{16}{70} + \left(\frac{2\,x}{l}\right)^4 - \frac{6}{5}\left(\frac{2\,x}{l}\right)^5 + \frac{1}{2}\left(\frac{2\,x}{l}\right)^6 - \frac{1}{14}\left(\frac{2\,x}{l}\right)^7\right]$ are plotted in Fig. 5. Both are close to a parabola. It may be assumed, therefore, that the torque consists of a component having a straight-line variation from a maximum of $\frac{w\,l}{2}\,e_1$ at the support to zero at midspan and a component having a parabolic variation from a maximum of $\frac{2}{3}\,w\,e_2\,\beta_o\,\frac{l}{2} + \frac{16}{70}\,\frac{w^2\,l^5}{64\,E\,I_y} = \frac{w\,l\,e_1}{2}\left(0.66\,e_2 + 0.00715\,\frac{w\,l^4}{E\,I_y}\right)\frac{\beta_o}{e_1}$ at the support to zero at midspan. The maximum angle of twist at midspan may then be computed as follows:

$$\beta_o = C\,R_u\,\frac{w\,l}{2}\,e_1 + C\,R_p\,\frac{w\,l}{2}\,e_1\left(0.66\,e_2 + 0.00715\,\frac{w\,l^4}{E\,I_y}\right)\frac{\beta_o}{e_1} = C\,R_u\,\frac{w\,l}{2}$$
$$\times\,e_1\left[1 + \frac{R_p}{R_u}\left(0.66\,e_2 + 0.00715\,\frac{w\,l^4}{E\,I_y}\right)\frac{\beta_o}{e_1}\right] \quad \text{..(24}a\text{)}$$

Fig. 3 shows that the quantity $\frac{R_p}{R_u}$ is practically constant and equal to 1.27. Using this value, the expression for β_o may be written as

$$\beta_o = \frac{\psi_o}{1 - 0.85\,\frac{\psi_o}{e_1}\left(e_2 + 0.01075\,\frac{w\,l^4}{E\,I_y}\right)} \quad \text{............(24}b\text{)}$$

Eq. 24b is similar to Eq. 21 and gives the maximum angle of twist under combined bending and torsion computed on the assumption that the effect of deflections on moments is not negligible. The quantity ψ_o is the maximum angle of twist that would have been obtained for uniform load if torque alone were acting and the deflections were negligible; that is,

$$\psi_o = C\,R_u\,\frac{w\,l\,e_1}{2} \quad \text{......................(25)}$$

3. Normal Fiber Stresses Under Combined Bending and Torsion

The normal fiber stress is composed of three parts: (1) The primary bending stress under vertical loads neglecting torsion; (2) the stress due to lateral

bending of the beam as a whole; and (3) the stress due to the bending of one flange relative to the other. The primary bending stress (part 1) is expressed as

$$f_b = \frac{M_x}{S_x} \qquad (26)$$

The lateral bending moment (part 2) is $\beta\, M_x$ and its maximum $\left(\text{which occurs for } x = \frac{l}{2}\right)$ is $\beta_o\, M_x$. The bending stress is then $\beta_o \frac{M_x}{S_y}$. The sum of parts 1 and 2 is $\frac{M_x}{S_x} + \beta_o \frac{M_x}{S_y} = f_b \left(1 + \frac{S_x}{S_y} \beta_o\right)$. It was assumed that for a concentrated load at midspan the torque is constant and equal to $\frac{\beta_o}{C\, R_c} = \frac{\beta_o}{\psi_o} \frac{C\, R_c}{\psi_o} = \frac{W\, e_1}{2} \frac{\beta_o}{\psi_o}$. The flange stress is then $B\, Q_c \frac{W\, e_1}{2} \frac{\beta_o}{\psi_o}$, in which ψ_o is given by Eq. 22. In the case of a uniform load, it is assumed that the torque is divided into two parts. One part has a linear variation from $\frac{w\, l\, e_1}{2}$ at the support to zero at midspan. The second part has a parabolic variation from $\frac{w\, l\, e_1}{2} \left(0.66\, e_2 + 0.00715 \frac{w\, l^4}{E\, I_y}\right) \frac{\beta_o}{e_1}$ at the support to zero at midspan. The flange stress is then

$$f = \frac{w\, l\, e_1}{2} B\, Q_u \left[1 + \frac{1}{e_1} \frac{Q_p}{Q_u} \left(0.66\, e_2 + 0.00715 \frac{w\, l^4}{E\, I_y}\right) \beta_o\right] \qquad (27)$$

As $\frac{Q_p}{Q_u}$ is nearly equal to $\frac{R_p}{R_u}$ (part 1), the quantity in brackets may be taken equal to $\frac{\beta_o}{\psi_o}$, and therefore the flange stress is $B\, Q_u \frac{w\, l\, e_1}{2} \frac{\beta_o}{\psi_o}$, in which ψ_o is given by Eq. 25. For more accuracy, $\frac{Q_p}{Q_u}$ may be assumed to be different from $\frac{R_p}{R_u}$. In this case, the quantity to be used instead of $\frac{\beta_o}{\psi_o}$ is $1 + \frac{Q_p}{Q_u} \frac{R_u}{R_p} \left(\frac{\beta_o}{\psi} - 1\right)$.

If the torque were considered alone, and if the effect of deflections were neglected, the flange bending stress would be: For a concentrated load,

$$f_f = B\, Q_c \frac{W\, e_1}{2} \qquad (28a)$$

and, for a uniform load,

$$f_f = B\, Q_u \frac{w\, l\, e_1}{2} \qquad (28b)$$

Using Eqs. 28, the flange bending stress (computed on the assumption that there is both bending and torque and that the effect of deflections is not negligible) may be expressed as $f_f \frac{\beta_o}{\psi}$ for both the concentrated and the uniform

loads. The total normal stress may then be represented by the equation:

$$f = f_b \left(1 + \frac{S_x}{S_y} \beta_o\right) + f_f \frac{\beta_o}{\psi_o} \qquad (29)$$

for both cases of loading. The quantities f_b, f_f, ψ_o, and β_o, obviously, do not have the same value for the concentrated load as for the uniform load.

It should be noted that the total stress is not proportional to the load, since β_o as given by Eqs. 21 and 24*b* is not proportional to the load. Therefore, the factor of safety must be applied to the load rather than to the stress. Let n be the factor of safety with respect to the yield point stress f_y. The criterion for the safe load is then

$$f_y = n f_b \left(1 + \frac{S_x}{S_y} \beta_{on}\right) + n f_f \frac{\beta_{on}}{n \psi_o} \qquad (30)$$

in which β_{on} is computed from Eqs. 21 and 24*b* for a load equal to n times the design load; that is, for a concentrated load,

$$\beta_{on} = \frac{n \psi_o}{1 - \frac{n \psi_o}{e_1}\left(e_2 + 0.0188\, n \frac{W l^3}{E I_y}\right)} \qquad (31a)$$

and, for a uniform load,

$$\beta_{on} = \frac{n \psi_o}{1 - \frac{n \psi_o}{e_1} \times 0.85 \left(e_2 + 0.0175 \frac{n w l^4}{E I_y}\right)} \qquad (31b)$$

The angle of twist ψ is given by Eqs. 22 and 25 for the concentrated and the unform loads, respectively. Eq. 30 may be rewritten as follows:

$$\frac{f_b}{f'_b} + \frac{f_f}{f'_f} = 1 \qquad (32)$$

in which

$$f'_b = \frac{\frac{f_y}{n}}{1 + \frac{S_x}{S_y} \beta_{on}} \qquad (33a)$$

and

$$f'_f = f_y \frac{\psi_o}{\beta_{on}} \qquad (33b)$$

Eq. 32 is similar to the formula of the American Institute of Steel Construction defining the criterion for the design of members under combined compression and bending.[6]

Numerical Examples.—Investigate a 14**WF**74 beam, 20 ft long, simply supported and loaded at a point 2 in. from the plane of the web ($e_1 = 2$ in.).

[6] "Specification for the Design, Fabrication and Erection of Steel for Buildings," A.I.S.C., New York, N. Y., February, 1946, p. 10.

Assume a factor of safety $n = 1.67$ with respect to the yield point stress of 33,400 lb per sq in. The following constants are taken from the "Manual of Steel Construction:"[2] $a = 63.04$; $B = 0.355$; $C = 0.00000144$; and $\frac{l}{2\,a} = 1.90$. From Figs. 3 and 4, $R_c = 0.943$; $Q_c = 0.956$; $R_u = 0.580$; and $Q_u = 0.370$.

Example 1.—Assume a concentrated load of 22,000 lb at midspan applied at the top flange ($e_2 = 7$ in.). Substituting in appropriate equations, the solution yields:

Equation	Quantity
22	$\psi_o = 0.0299$
31*a*	$\beta_{on} = 0.066$
33*a*	$f'_b = 15{,}600$
33*b*	$f'_f = 15{,}100$
26	$f_b = 11{,}750$
28*a*	$f_f = 7{,}450$

Testing the foregoing results by Eq. 32: $\frac{f_b}{f'_b} + \frac{f_f}{f'_f} = \frac{11{,}750}{15{,}600} + \frac{7{,}450}{15{,}100} = 1.250.$

Example 2.—Assume a uniform load of 2,200 lb per foot of span ($w\,l = 44{,}000$ lb) applied at the top flange ($e_2 = 7$ in.). As in example 1, substituting in the appropriate equations, the solution yields

Equation	Quantity
25	$\psi_o = 0.0367$
31*b*	$\beta_{on} = 0.082$
33*a*	$f'_b = 14{,}800$
33*b*	$f'_f = 14{,}900$
26	$f_b = 11{,}750$
28*b*	$f_f = 5{,}790$

Again testing results by Eq. 32: $\frac{11{,}750}{14{,}800} + \frac{5{,}700}{14{,}900} = 1.183.$

4. Critical Loads

The critical load is reached when the angle of twist and the lateral deflection become very large. Since the lateral deflection is proportional to the angle of twist, it is sufficient to consider the angle of twist only. Eqs. 21 and 24*b* show that the critical load is determined by: For a concentrated load,

$$C\,R_c\,\frac{W}{2}\left(e_2 + 0.0188\,\frac{W\,l^3}{E\,I_y}\right) = 1 \qquad (34a)$$

and, for a uniform load,

$$0.85\,C\,R_u\,\frac{w\,l}{2}\left(e_2 + 0.01075\,\frac{w\,l^4}{E\,I_y}\right) = 1 \qquad (34b)$$

In computing the stresses (Section 3), β_{on} was determined for n times the design load. The value of β_{on} thus found shows whether n times the design

load approaches the critical value. Therefore, no additional calculations for critical loads are required.

If the eccentricity e_2 is zero, Eqs. 34 may be rewritten as follows: For a concentrated load,

$$0.0094\, R_c \frac{a}{l} \frac{l^4}{E_s\, \kappa\, E\, I_y} W^2 = 1 \qquad (35a)$$

and, for a distributed load,

$$0.00458\, R_u \frac{a}{l} \frac{l^4}{E_s\, \kappa\, E\, I_y} (w\, l^2) = 1 \qquad (35b)$$

Therefore, the critical load becomes: For a concentrated load,

$$W = m \frac{\sqrt{E_s\, \kappa\, E\, I_y}}{l^2} \qquad (36a)$$

and, for a distributed load,

$$w\, l = m \frac{\sqrt{E_s\, \kappa\, E\, I_y}}{l^2} \qquad (36b)$$

in which the coefficient m is: For a concentrated load,

$$m = 10.3 \sqrt{\frac{l}{a\, R_c}} \qquad (37a)$$

and, for a distributed load,

$$m = 14.8 \sqrt{\frac{l}{a\, R_u}} \qquad (37b)$$

TABLE 2.—Comparison of Values of m

$\left(\frac{l}{a}\right)^2$	$\frac{l}{2\,a}$	Concentrated Load		Uniform Load	
		Timoshenko	Eq. 37a	Timoshenko	Eq. 37b
0.4	0.316	86.4	82.5	143	....
4.0	1.000	31.9	30.2	53.0	54.3
8.0	1.415	25.6	24.0	42.6	43.7
16.0	2.000	21.8	20.4	36.3	37.2
24.0	2.450	20.3	19.0	33.8	34.3
32.0	2.828	19.6	18.3	32.6	33.5
48.0	3.465	18.8	17.4	31.5	32.2

Eqs. 36 are the same as those presented by S. Timoshenko in 1936.[7] The comparison of the numerical values of coefficients m in Table 2 shows that the difference between the two sets of coefficients is not large in spite of the fact that they were established by entirely different methods.

Conclusions

Equations for normal stress in I-beams loaded in combined bending and torsion are developed in this paper. These equations take into account the interaction between bending and torsion and the effect of the deflections on moments. It is found that the allowable stress in bending is affected by the torque and that the primary bending affects the allowable stress for torsional flange bending. The numerical coefficients are computed for the values of $\frac{l}{2\,a}$ included between 0 and 3 since in most practical cases $\frac{l}{2\,a}$ falls between these limits.[2]

[7] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, pp. 267 and 268, paragraph 52.

The shearing stresses can be obtained from Eqs. 7. The principal normal stress and the maximum shearing stress resulting from the combination of the normal fiber stress and shear will usually occur in a diagonal direction. The analysis of the principal stresses, however, is beyond the scope of this paper.

The main objective of this study is to analyze the stresses under the conditions listed in the "Introduction," rather than to develop practical design formulas. The expressions obtained, however, although somewhat lengthy, are of the type that can be rather conveniently used in design.

Acknowledgment

In preparing this paper the published works of several writers were found helpful, although they have not been cited directly. Among the writers included in this list are: D. H. Young[8] and George Winter,[9,10] N. M. Newmark,[11] and C. R. Young,[12,13] Members, ASCE.

APPENDIX. NOTATION

The following letter symbols conform essentially with "American Standard Letter Symbols for Structural Analysis" (ASA—Z10.8—1949) prepared by a sectional committee of the American Standards Association, with ASCE participation, and approved by the Association in 1949. Discussers are requested to adapt their comments to the symbols in the paper.

A_1 = integration constant (see B_1 and C_1);
a = torsional bending constant;
B = torsional stress constant;
B_1 = integration constant (see A_1 and C_1);
b = width of flange;
C = torsional twist constant;
C_1 = integration constant (see A_1 and B_1);
d = distance between flange centroids;
E = elasticity; modulus of elasticity in tension; E_s denotes the modulus of elasticity in shear;
e = eccentricity in relation to the plane of the web:
  e_1 = eccentricity normal to the web;
  e_2 = eccentricity parallel to the web;
f = normal fiber stress:
  f' = allowable value of f;
  f_b = stress in bending;

[8] "Rational Design of Steel Columns," by D. H. Young, *Transactions*, ASCE, Vol. 101, 1936, p. 422.

[9] "Lateral Stability of Unsymmetrical I-Beams and Trusses in Bending," by George Winter, *ibid.*, Vol. 108, 1943, p. 247.

[10] "Strength of Slender Beams," by George Winter, *ibid.*, Vol. 110, 1944, p. 1321.

[11] "Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," by N. M. Newmark, *ibid.*, Vol. 108, 1943, p. 1161.

[12] *Engineering News-Record*, November 27, 1924, p. 882.

[13] *Ibid.*, December 25, 1924, p. 1047.

f_f = stress in torsional bending of one flange with respect to the other;
f_y = yield point stress;
H = total flange shear force;
I = moment of inertia about the principal axis:
I_x = moment of inertia normal to the web;
I_y = moment of inertia parallel to the web;
l = length of span;
M = bending moment about a principal axis:
M_x = moment normal to the web;
M_y = moment parallel to the web;
m = a coefficient defined by Eqs. 37;
n = a number (factor of safety with respect to yield point);
Q = normal stress function, subscripts c, u, p, and s denoting case 1, case 2, case 3, and case 4, respectively;
R = torsional twist function, subscripts c, u, p, and s denoting case 1, case 2, case 3, and case 4, respectively;
S = section modulus about the principal axis, subscripts x and y denoting axes normal to both and parallel to web, respectively;
T = torque:
T' = part of the torque resisted by the torsional constant;
T'' = part of the torque resisted by the lateral bending of flanges;
T_x = torque applied at an abscissa distance x from the origin of the axes;
W = concentrated load (weight) at midspan;
w = uniform load per unit length of span;
X_1 = a parameter expressing a particular solution depending on the nature of the torque;
x = coordinate with y;
y = lateral deflection of a beam:
y_c = maximum value of y;
y_1 = value of y at a distance x_1;
β = angle of twist resulting from the combined bending and torsion computed on the assumption that the effect of deflection is not negligible:
β_n = value of β computed n times the design load;
β_o = maximum value of β;
κ = torsional constant; and
ψ = angle of twist resulting from applied torque alone, computed on the assumption that the effect of deflections is negligible; the maximum value of ψ is denoted by ψ_o.

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